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A NOTE ON MILESTONES OF LABELING OF GRAPHS

Bipinkumar V
Department of Mathematics, Sree Narayana College,
Cherthala, Alappuzha, Kerala, India.
vbkumarsncmaths@gmail.com

Abstract

A graph labeling is a map that carries graph elements such as vertices or edges to numbers (usually to the positive or non-negative integers). Some labeling use the vertex-set alone, or the edge-set alone, and we shall call them vertex-labeling and edge-labeling respectively. Numerous labeling techniques have been defined and a lot of surveys of labeling techniques have been carried out so far. This paper is an attempt to depict a journey through some of the graph labeling techniques which may be mentioned as milestones in the history of labeling and have become inspiration for many other labeling techniques.

Keywords: Labeling of graphs, Graceful graphs, Set-indexers of graphs.

Introduction

The theory of graphs is a very popular area of discrete mathematics having numerous theoretical developments and countless applications to engineering and other branches of sciences.Graph theory is considered to have begun in 1736 with the publication of Euler's solution of Konigsberg Bridge Problem.Any mathematical object involving points and connections between them may be called a Graph. Labeling is a term used in technical sense for naming objects using symbolic format drawn from any universe of discourse such as the set of numbers, algebraic groups or the power set 2^X of a 'ground set' X. The objects requiring labeling could come from a variety of fields of human interest such as chemical elements, radio antennae, spectral bands and plant/animal species. Further, categorization of objects based on certain clustering rules might lead to derived labels from the labels of objects in each cluster; for instance labels a and b of two individual elements in a set {A, B} could be used to derive a labeling for the set in a way that could reflect a relational combination of the labels a and b. To be specific, A and B are assigned labels a, b from an algebraic group, whence the set {A, B} is assigned the label a*b where * is the group operation. Such assignments are generally motivated by a need of optimization on the number of symbols used to label the entire discrete structure so that the structure could be effectively encoded for handling its computerized analysis.

Preliminaries

The basic definitions of graph theory can be found in [4] and [7]. A graph Gis an ordered pair (V, E) where V is a set of elements called vertices and E is a set of unordered pairs of distinct vertices from V called edges. We say two vertices u and v are adjacent if they are connected by an edge. The set of adjacent vertices of a vertex u is denoted as N (u), and it is also called the set of neighbors of u. The degree of a vertex u is d (u) =|N (u)|, the number of neighbors of u. For a given graph G, when the vertex set and the edge set are not given explicitly, we refer to them as V (G) and E (G) respectively.

A subgraph H of G is a graph such that V (H) ⊆V (G) and E (H) ⊆E (G). For a graphG=(V, E) and a subset W⊆V, the subgraph of G induced by W, denoted as G[W], is the graph H=(W, F) such that, for all u, v∈W, if (u, v)∈E, then (u, v)∈F. We say H is an induced subgraph of G. Equivalently, we can define subgraphs and induced sub graphs in terms of deletion of vertices and edges: H is an induced subgraph of G if it is obtained by deletion of vertices, and H is a subgraph of G if it is obtained by deletion of vertices and edges.

A graph G is said to be connected, if every pair of vertices is connected by a path. If there is exactly one path connecting each pair of vertices, we say G is a tree. Equivalently, a tree is a connected graph with (n-1) edges. A path graph P_n is a connected graph on n vertices such that degree of each vertex is less than or equal to two. A cycle graph C_n is a connected graph on n vertices such that every vertex has degree exactly two.

A complete graph K_n is a graph with n vertices such that every vertex is adjacent to all the others. A bipartite graph G= (V, E) is a graph such that there exists a partition P= (A, B) of V such that every edge of G connects a vertex in A to one in B. Equivalently, G is said to be bipartite if A and B are independent sets. The bipartite graph is also denoted as G= (A, B, E) where E is the edge set. The join of two graphs G_1= (V_1, E_1) and G_2= (V_2, E_2) with disjoint vertex sets V_1 and $V_2$is the graph G= (V, E) such that $V=V_1\cup V_2$ and $E=E_1\cup E_2\cup\{(u, v): u\in V_1, v\in V_2\}$, that is, G is obtained by connecting every vertex of G_1 to every vertex of G_2.

Labeling of a graph

A Graph labelingis an assignment of finite subset of set of integers to the vertices or edges or both subject to certain conditions.'Graph labeling' as an independent notion using numbers was first introduced by A. Rosa [8] in 1967.B. D. Acharya [1]associated an arbitrary non-empty set to the graphs and laid foundations for the set-valuations of graphs.Later, numerous graph and digraph labeling techniques were introduced and have been remarkably surveyed by Joseph A Gallian [5]. We shall define two labeling of the same graph to be equivalent if one can be transformed into the other by an automorphism of the graph. In general, graph labeling, where the basic elements (i.e., vertices and/or edges) of a graph are assigned elements of a given set or subsets of a nonempty 'ground set' subject to certain given conditions, has often been motivated by a lot of practical situations.

Graceful graphs

Even though the study of graceful graphs and graceful labeling methods was introduced by Rosa, the term graceful graph was used first by Golomb[6] in 1972. Rosa defined a β-valuation of a (p, q)-graph G as an injection *f* from the vertices of G to the set {0, 1… q-1} such that, when each edge (x, y) is assigned the label $|f(x)-f(y)|$, the resulting edge labels are all distinct. The β-valuations of a graph originated as a means of attacking the conjecture of Ringel.A few years later, Golombrenamed β-labeling as graceful labeling as it is known today. Many problems of graph theory consist in finding a vertex or an edge labeling for a graph satisfying certain properties. In a graceful labeling of a graph G the resulting edge labels must be distinct and take values 1, 2… q.

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That is, the vertices are labeled with distinct numbers chosen from 0 to m, where m is the number of edges, such that each edge is labeled with the absolute difference of the labels of its end vertices and it is unique in the graph. Rosa showed that if every tree is graceful, then Ringel's conjecture holds. Since then, researchers have been trying to prove Ringel's conjecture through the Graceful Tree Conjecture, which claims that every tree is graceful. However, graceful graphs gained their own merit of study over the years of research.

A graph is said to be graceful if the numbers used to label its vertices are distinct values of the set {0,1,2,3 . . . m} and the edge labels are distinct values of the set {1,2,3 . . . m} where the edge labels are absolute values of the difference of vertex labels. Rosa defined the α-labelingof a graph as a graceful labeling with the additional property that there exists an integer k so that for every edge (x, y) either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. The graphs with α-labeling have proved to be useful in the development of the theory of graph decompositions.Acharya proved that every graph can be embedded as an induced subgraph of a graceful graph [2] and a connected graph can be embedded as an induced subgraph of a graceful connected graph.

If a (p, q)-graph G has a set-graceful labeling with respect to a set X of cardinality $m \geq 2$, then there exists a partition of the vertex set V(G) into two nonempty sets A and B such that the number of edges joining the vertices of A with those of B is exactly 2^{m-1}.It is also known that a graph G with q edges is k-graceful if there exists a labeling f from the vertices of G to the set {0,1,2, q+k-1} such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is the set {k, k+1…, q+k-1}. Thus a graceful labeling of a graph G is a vertex labeling f: V→[0, m] such that the function f is injective and the induced edge labeling $f\gamma$:E→[1, m] defined by $f\gamma(u, v)=|f(u)-f(v)|$ is also injective. If a graph G admits a graceful labeling, then we say G is a graceful graph. Although the theory of graceful graphs has been studied for a very long time-span, not many general results are known about graceful labeling. Most of the results are about asserting the gracefulness of a graph-class since it suffices to show a graceful labeling for each graph in the class. The characterization of graceful graphs has always remained as an active area for the researchers.

The problem of graceful labeling of a graph

The study of graceful labeling of a graph has always been a very prolific area of research in graph theory. The graceful labeling problem is to determine which graphs are graceful. Proving a graph G is or is not graceful involves either producing a graceful labeling of G or showing that G does not admit a graceful labeling. The graceful labeling of graphs is perceived to be a primarily theoretical topic in the field of graph theory.The gracefully labelled graphs often serve as models in a wide range of applications.These applications include coding theory and communication network addressing. Bloom and Golomb give a detailed account of some of the important applications of gracefully labelled graphs. That 'all trees are graceful' is a long-standing conjecture known as the "Ringel–Kotzig Conjecture". A characteristic change from assigning numbers to the basic elements of a given graphG was made by suggesting to consider set-valued functions, and these changes were motivated by certain considerations in social psychology. Interpersonal relationships depend on personal attitudes of the individuals in any social group. When opinions are expressed by the individuals to others in the group, the types of interpersonal interactions get affirmed and/or modified. On the other hand, such affirmations and/or modifications in various types of interpersonal interaction in the group could induce change in the attitudes of the persons in the group. Actually, this phenomenon has motivated a study of total set-valuations or assignment of subsets of a given set to the basic elements of a given graph with a variety of constraints motivated either by theoretical or by practical considerations. Thus a set-valuation of a graph G=(V, E) is simply an assignment of elements of the power set 2^X of a given nonempty 'ground set' X to the basic elements of G; having a variety of origins.

A Set-indexer of a graph

A Set-indexer of a graph G=(V, E) is an injective 'vertex set-valuation' $f : V(G) \to 2^X$ such that the induced edge set-valuationon the edges (u, v) of G defined by $f(u) \oplus f(v)$, $\forall (u,v) \in E(G)$ is also injective, where '$\oplus$' denotes the operation of taking the symmetric difference of the subsets of X. The symmetric difference of two sets A and B can be the emptysetonly if A and B are equal.

Thus assigning an injective vertex labeling rules out the possibility of getting the emptyset as an edgelabel. It is proved that every graph has a set-indexer [1]. Here X is called an indexing set of G. A graph can have many indexing sets and the minimum of the cardinalities of these indexing sets is called the set indexing number of G.A graph G is said to be set-graceful if there exists a set X and a set-indexer f:V(G)→2^Xsuch that f(E(G))=2^X-{φ}.The characterization of set graceful graphs is still an open problem in this area.

There are results analogue of well-known properties of arbitrary networks like Kirchhoff's Voltage Law (KVL); the analogy could be seen by treating P(X) as an additive 'voltage group' where the 'addition' is the binary operation of taking symmetric difference between any two subsets of X.

Topological set-indexers

A set-indexer f:V(G)→2^X such that the family f(V(G))={f(u):u∈V(G)} is a topology on X is calleda topological set-indexer (or, a T-set-indexer) [2]. Here G= (V, E) is a graph and X is an underlying set. This definition established a link between graph theory and point-set topology.It is proved that corresponding to every graph G there exists a topological set-indexer. We can relate a graph G to different topological structures. In 1967, J. W.Evans proved that there is a one to one correspondence between the set of all topologies with n points and the set of all transitive digraphs with n points. The labeling of vertices and edges of a graph G subject to certain conditions have been often motivated by their utility in various applied fields and their intrinsic mathematical interest. The well-known Four Colour theorem was originated in 1853 and remained as Four Colour Conjecture for more than 150 years till it was solved in 1977. Later following many illustrious works on β-valuations, there followed a lot of other ways of labeling such as arithmetic labeling,felicitous labeling, elegant labeling, sequential labeling etc. The field of set-valuations and set-graceful labeling is an active area of research providing splendid open problems and new directions in the theory of set-valuations of graphs.

Topogenic set-indexers of a graph

A set-indexer f of a graph G=(V, E) is calledtopogenic [3], if the family $f(V(G)) \cup f_L(E(G))$ is a topology on X, where $f(V(G))=\{f(u):u\in V(G)\}$ and $f_L(E(G))=\{f_L(e):e\in E(G)\}$. In particular, if $f(V(G)) \cup f_L(E(G))=P(X)$, then f is called a graceful topogenic set-indexer. Analogous to these set-indexers, a lot of different set indexing methods wereevolved and that is why these set-indexers are considered as milestones in the history of labeling of graphs.The comparison between various types of set-valuations of graphs and their applications are excellently surveyed by B. D. Acharya and K.A. Germina in [3].

Conclusion

The field of labeling of graphs have been an active area of research ever since its introduction. Based on the labeling techniques discussed here, numerous labeling techniques have been defined and a lot of surveys of labeling techniques have been carried out so far. Still, a lot of research scholars are working in the area.The concept of labeling has been adopted and visualized by many other related branches of mathematics. Deriving new labeling techniques and finding applications of the labeling techniques in real life situations are active open research problems for researchers.

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